More Homogeneous Linear System Examples

1. Solve $\bar{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = 0$$

$$(3-r)(-2-r)+4=0$$
$$-6-3r+2r+r^2+4=0$$
$$r^2-r-2=0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{1 \pm \sqrt{1+8}}{2}$$
$$= \frac{1 \pm 3}{2}$$
$$= -1 \text{ or } 2$$

$(A-rI)\bar{z} = \bar{0}$

when $r = -1$

$$\begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4z_1 - 2z_2 = 0$$
$$2z_1 - z_2 = 0 \leftarrow \text{Redundant}$$

$2z_1 - z_2 = 0$
$2z_1 = z_2$
Let $z_1 = 1$. $z_2 = 2$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

When $r = 2$

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - 2z_2 = 0$
$2z_1 - 4z_2 = 0$ ← Redundant

$z_1 = 2z_2$

Let $z_1 = 1$. $z_2 = 1/2$

$$\bar{x} = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

2. Solve $\bar{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & -2 \\ 3 & -4-r \end{vmatrix} = 0$$
$(1-r)(-4-r) + 6 = 0$
$-4 - r + 4r + r^2 + 6 = 0$
$r^2 + 3r + 2 = 0$
$(r+2)(r+1) = 0$
$r_1 = -2 \quad r_2 = -1$

$(A - rI)\bar{z} = \bar{0}$

When $r = -2$

$$\begin{bmatrix} 1+2 & -2 \\ 3 & -4+2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - 2z_2 = 0$
$3z_1 - 2z_2 = 0 \leftarrow$ Redundant

$3z_1 = 2z_2$
$\frac{3}{2} z_1 = z_2$
Let $z_1 = 1$. $z_2 = 3/2$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$$

When $r = -1$

$$\begin{bmatrix} 1+1 & -2 \\ 3 & -4+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2z_1 - 2z_2 = 0$
$3z_1 - 3z_2 = 0 \leftarrow$ Redundant

$z_1 = z_2$
Let $z_1 = 1$. $z_2 = 1$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. Solve $\bar{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0$$

$(2-r)(-2-r)+3=0$
$-4-2r+2r+r^2+3=0$
$r^2-1=0$
$r^2=1$
$r_1=1, \; r_2=-1$

$(A-rI)\bar{z}=\bar{0}$
when $r=1$

$$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$
$3z_1 - 3z_2 = 0$ ← Redundant

$z_1 = z_2$
let $z_1 = 1$. $z_2 = 1$.

$$\bar{z}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $r = -1$

$$\begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$
$3z_1 - z_2 = 0$ ← Redundant

$3z_1 = z_2$
Let $z_1 = 1$. $z_2 = 3$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\bar{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

4. Solve $\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \bar{x}$, $\bar{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Soln:

$$\begin{vmatrix} 5-r & -1 \\ 3 & 1-r \end{vmatrix} = 0$$

$(5-r)(1-r) + 3 = 0$
$5 - 5r - r + r^2 + 3 = 0$
$r^2 - 6r + 8 = 0$
$(r-2)(r-4) = 0$
$r_1 = 2$, $r_2 = 4$

$(A - rI)\bar{z} = \bar{0}$

when $r=2$

$$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$

$3z_1 = z_2$

Let $z_1 = 1$. $z_2 = 3$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

When $r = 4$

$$\begin{bmatrix} 5-4 & -1 \\ 3 & 1-4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$c_1 + c_2 = 2$

$3c_1 + c_2 = -1$

$-2C_1 = 3 \rightarrow C_1 = -\frac{3}{2}$

$C_2 = 2 - C_1$

$= 2 + 3/2$

$= \frac{7}{2}$

$$\vec{x} = \frac{-3}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{7}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. Solve $\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$

Soln:

$$\begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = 0$$

$(1+r)^2 + 4 = 0$

$r^2 + 2r + 1 + 4 = 0$

$r^2 + 2r + 5 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$(A - rI)\vec{z} = \vec{0}$

when $r = -1 + 2i$

$$\begin{bmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-1+1-2i)z_1 - 4z_2 = 0$

Note: In $A-rI$, even for complex numbers, the 2 rows are still linearly dependent.

$-2iz_1 = 4z_2$

$-iz_1 = 2z_2$

$\frac{-iz_1}{2} = z_2$

Let $z_1 = 2$. $z_2 = -i$

$$\overline{z^1} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Recall that $\vec{x} = e^{rt}\vec{z}$.

In this case, because we have complex eigenvalues, $r = \lambda + iu$.

$$\vec{x} = e^{\lambda t + iut}\vec{z}$$

$$= e^{\lambda t} \cdot \underbrace{e^{iut}}_{} \cdot \vec{z}$$

Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Hence, $\vec{x} = e^{\lambda t}(\cos(ut) + i\sin(ut))\vec{z}$.

In this example, $\lambda = -1$ and $u = 2$.

$$\overline{x} = e^{-t}(\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$= e^{-t}\cos(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} +$$

$$i\left(e^{-t}\cos(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + e^{-t}\sin(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

$$\bar{x} = c_1\left(e^{-t}\cos(2t)\begin{bmatrix}2\\0\end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right) +$$
$$c_2\left(e^{-t}\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix} + e^{-t}\sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$$

6. Solve $\bar{x}' = \begin{bmatrix}2 & -5\\1 & -2\end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix}2-r & -5\\1 & -2-r\end{vmatrix} = 0$$
$(2-r)(-2-r)+5=0$
$-4-2r+2r+r^2+5=0$
$r^2+1=0$
$r^2=-1$
$r=\pm i$

$(A-rI)\bar{z}=\bar{0}$
when $r=i$

$$\begin{bmatrix}2-i & -5\\1 & -2-i\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$(2-i)z_1 - 5z_2 = 0$
$(2-i)z_1 = 5z_2$
$\frac{(2-i)z_1}{5} = z_2$
Let $z_1 = 5$. $z_2 = 2-i$

$$\bar{Z}' = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\bar{x} = e^{rt} \bar{Z}$$

$$e^{rt} = e^{it}$$
$$= \cos(t) + i\sin(t)$$

$$(\cos(t) + i\sin(t)) \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= \cos(t) \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + i \left(\sin(t) \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$\bar{x} = C_1 \left(\cos(t) \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) +$$

$$C_2 \left(\sin(t) \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

7. Solve $\bar{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0$$

$(1-r)(-3-r)+5=0$

$-3-r+3r+r^2+5=0$

$r^2+2r+2=0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$= -1 \pm i \rightarrow r_1 = -1+i, \; r_2 = -1-i$

$(A-rI)\bar{z} = \bar{0}$

when $r=-1+i$

$$\begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-(-1+i))z_1 - z_2 = 0$

$(1+1-i)z_1 = z_2$

$(2-i)z_1 = z_2$

Let $z_1 = 1$, $z_2 = 2-i$

$$\bar{z}' = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\bar{x} = e^{rt}\,\bar{z}$$

$$= e^{(-1+i)t}\left(\begin{bmatrix}1\\2\end{bmatrix} + i\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

$$= e^{-t}\cdot e^{it}\cdot\left(\begin{bmatrix}1\\2\end{bmatrix} + i\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

$$e^{it} = \cos(t) + i\sin(t)$$

$$(\cos(t)+i\sin(t))\left(\begin{bmatrix}1\\2\end{bmatrix} + i\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

$$= \cos(t)\begin{bmatrix}1\\2\end{bmatrix} - \sin(t)\begin{bmatrix}0\\-1\end{bmatrix} +$$

$$i\left(\sin(t)\begin{bmatrix}1\\2\end{bmatrix} + \cos(t)\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

$$\bar{x} = C_1 e^{-t}\left(\cos(t)\begin{bmatrix}1\\2\end{bmatrix} - \sin(t)\begin{bmatrix}0\\-1\end{bmatrix}\right) +$$

$$C_2 e^{-t}\left(\sin(t)\begin{bmatrix}1\\2\end{bmatrix} + \cos(t)\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

8 Solve $\bar{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \bar{x}$

Soln:

$\begin{vmatrix} 3-r & -4 \\ 1 & -1-r \end{vmatrix} = 0$

$(3-r)(-1-r)+4=0$

$-3-3r+r+r^2+4=0$

$r^2-2r+1=0$

$(r-1)^2=0$

$r=1$

$(A-rI)\bar{z}=\bar{0}$

when $r=1$

$\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2z_1 - 4z_2 = 0$

$z_1 - 2z_2 = 0$

$\frac{z_1}{2} = z_2$

Let $z_1 = 2$. $z_2 = 1$.

$\overline{z^1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Because we have repeated roots,
$\bar{x} = c_1 e^{rt} \bar{z} + c_2 (t e^{rt} \bar{z} + e^{rt} \bar{p})$, where
$\bar{p}$ is a generalized eigenvector.

$(A - rI)\bar{p} = \bar{z}$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$2p_1 - 4p_2 = 2$
$p_1 - 2p_2 = 1$

Let $p_2 = 0$. $p_1 = 1$.

$$\bar{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, $\bar{x} = c_1 \left(e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) + c_2 \left(t e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

9. Solve $\overline{x'} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = 0$$

$(4-r)(-4-r)+16=0$

$-16-4r+4r+r^2+16=0$

$r^2=0$

$r=0$

$(A-rI)\overline{z}=\overline{0}$

when $r=0$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 2z_2 = 0$

$2z_1 - z_2 = 0$

$2z_1 = z_2$

Let $z_1 = 1$. $z_2 = 2$.

$$\overline{z'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\overline{x} = c_1 e^{rt}\overline{z} + c_2\left(te^{rt}\overline{z} + e^{rt}\overline{p}\right)$$

$(A - rI)\bar{p} = \bar{z}$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$2p_1 - p_2 = \frac{1}{2}$

Let $p_1 = 0$. $p_2 = \frac{-1}{2}$

$$\bar{p} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

$$\bar{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right)$$